

# Nonlinear resonant tunneling of Bose-Einstein condensates in tilted optical lattices

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We study the tunneling decay of a Bose-Einstein condensate out of tilted optical lattices within the mean-field approximation. We introduce a novel method to calculate also excited resonance eigenstates of the Gross-Pitaevskii equation, based on a grid relaxation procedure with complex absorbing potentials. This algorithm works efficiently in a wide range of parameters where established methods fail. It allows us to study the effects of the nonlinearity in detail in the regime of resonant tunneling, where the decay rate is enhanced by resonant coupling to excited unstable states.

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## I. INTRODUCTION

The dynamics of a quantum particle in a periodic potential subject to an external force is one of the central problems in solid state physics. In the field free case all eigenstates are delocalized over the lattices, leading to transport [1, 2]. The application of a constant force leads to a localization of the eigenstates such that transport is suppressed contrary to our intuition [3–6]. Instead, the quantum particle performs the celebrated Bloch oscillations, and eventually decays by repeated Zener tunneling to higher Bloch bands [7–17]. The most detailed studies of Bloch oscillations and decay have been carried out with ultracold atoms trapped in optical lattices. These systems are particularly appealing, because the dynamics of the atoms can be recorded in situ and all parameters can be tuned precisely over a wide range. The external force can be induced by gravity [7], magnetic gradient fields [8] or by accelerating the lattice [9–13]. Decay in strong fields manifests itself in the pulsed output of coherent matter waves. The dynamics is even more interesting when the atoms undergo Bose-Einstein condensation and interactions have to be taken into account. For low temperature and high densities, the dynamics of the atoms can be described by the celebrated Gross-Pitaevskii equation (GPE) with astonishing accuracy [18]. In this treatment, interactions are incorporated by a nonlinear mean-field potential, which is proportional to the condensate density. The nonlinearity of the equation alters the dynamics and in particular the decay substantially. Interactions can lead to a damping of Bloch oscillations [19], asymmetric Landau-Zener tunneling [10, 20, 21], or a bistability of resonance curves [22–24].

Here we study the resonance eigenstates of the GPE

$$\left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) + Fx + g|\psi(x)|^2 \right) \psi(x) = (\mu - i\Gamma/2)\psi(x) \quad (1)$$

with a periodic potential  $V(x+d) = V(x)$  and a static

force  $F > 0$ , which is known as a Wannier-Stark (WS) potential. The imaginary part  $\Gamma$  of the eigenenergy gives the decay rate of the condensate. In the following we use scaled units with  $\hbar = m = 1$  and we consider a cosine potential  $V(x) = \cos(x)$  unless otherwise stated. A comprehensive review of the localized eigenstates, the WS resonances, can be found in [16].

In this article we introduce a new algorithm for the computation of nonlinear resonance states based on a grid relaxation method with a complex absorbing potential (CAP). This algorithm converges in a wide parameter range and is applicable even to situations of many degenerate energy levels, such as the WS system at resonance condition (see below). It is thus capable to describe genuine nonlinear phenomena such as bistability, which pose a major difficulty to other methods as for instance nonlinear complex scaling (CS) [25–29]. In addition, it is more efficient and easier to implement and, unlike previous methods, is not restricted to ground state calculations but can also compute excited states. Note that our approach differs from the CAP method used in [26, 30] because the latter does not use a grid relaxation but relies on a basis set expansion. Though such expansions work well for simple single well potentials, they cannot easily handle complicated problems like the Wannier-Stark system studied in the present paper, which requires the use of as much as 500 basis states even in the linear (noninteracting) case [31]. Our method is applied to study the decay of a Bose-Einstein condensate in the strongly nonlinear regime. Nonlinear effects are crucial in the regime of resonantly enhanced tunneling (RET). In this case a metastable WS resonance becomes energetically degenerate with an excited, less stable state, which can increase the decay rate by orders of magnitude. This phenomenon is most pronounced in deep optical lattices and has been studied systematically for the linear case in [15, 16]. The nonlinearity shifts the resonance and eventually bends the resonance peak leading to a bistable behavior.

## II. COMPUTATIONAL METHOD

Linear WS resonances can be efficiently calculated with the truncated shift operator technique introduced in [31]. In the nonlinear case, the method of CS has been applied [25, 27–29]. Though satisfactory from a conceptual point of view, this method has several drawbacks. The implementation is complicated as it requires switching between different basis sets as well as different time propagation methods. Furthermore, the calculation of excited states is highly non-trivial, as the method relies on an imaginary time propagation, and the convergence is quite slow, especially for weak fields and close to energetic degeneracies as present in the RET condition [27, 28]. As an alternative, we propose a method based on complex absorbing potentials (CAP) performed on a finite grid  $[x_-, x_+]$  in real space. We assume that the resonance wave function is mainly localized in the interval  $[x_\ell, x_r]$  with  $x_- < x_\ell < x_r < x_+$  and fix the normalization as  $\int_{x_\ell}^{x_r} |\psi(x)|^2 dx = 1$ . For  $x \rightarrow -\infty$ , we apply a CAP of the type

$$V_{\text{CAP}} \propto \begin{cases} -i(x/x_-)^{10} & x < x_- \\ 0 & x > x_- \end{cases}, \quad (2)$$

which only modifies the wave function in the vicinity of the grid boundary  $x_-$  making it square integrable. For  $x \rightarrow +\infty$ , the wavefunction rapidly converges to zero, so that no further CAP is needed on this side. The boundary conditions for the wave function read

$$\psi(x_-) = 0, \quad \psi(x_+) = 0, \quad \psi'(x_+) = C, \quad (3)$$

where the last condition is used to control the normalization. The algorithm starts from the linear case  $g = 0$ , for which all WS resonances can be computed efficiently [31]. Nonlinear WS resonances in different bands are calculated by choosing a different initial guess. The nonlinearity is then increased gradually, using the previous result as initial guess for a standard boundary value problem (BVP) solver, e.g. the MATLAB -function `bvp4c`. Applying the BVP solver changes the normalization of  $\psi$ , such that the parameter  $C$  has to be adjusted according to

$$C \rightarrow C / \left( \int_{x_\ell}^{x_r} |\psi(x)|^2 dx \right)^{1/2}. \quad (4)$$

This is repeated until the normalization converges to unity. Then the nonlinearity is increased by one step. To demonstrate the validity of this algorithm we compare the calculated decay rates for a cosine potential for several parameters to complex scaling results, which themselves were tested against a direct time propagation in Ref. [27]. The values summarized in Tab. I show an excellent agreement over the entire parameter range. Residual numerical errors are very small; they can mainly be attributed to the limited computation time for the CS method and reflections of the matter wave at the CAP. For a further discussion of CAPs in the simulation of few boson systems, see [32] and references therein.

$g$	$F$	$\Gamma_{\text{CS}}$	$\Gamma_{\text{CAP}}$
0	0.5	$1.941 \times 10^{-2}$	$1.941 \times 10^{-2}$
0.1	0.5	$2.180 \times 10^{-2}$	$2.180 \times 10^{-2}$
0	0.25	$7.2 \times 10^{-4}$	$7.104 \times 10^{-4}$
0.1	0.25	$8.4 \times 10^{-4}$	$8.346 \times 10^{-4}$
0.2	0.25	$9.7 \times 10^{-4}$	$9.688 \times 10^{-4}$
0.25	0.25	$1.04 \times 10^{-3}$	$1.041 \times 10^{-3}$
0.5	0.25	$1.48 \times 10^{-3}$	$1.476 \times 10^{-3}$
0.2	0.15	$2.9 \times 10^{-5}$	$2.832 \times 10^{-5}$
0.2	0.13125	$5.7 \times 10^{-5}$	$5.600 \times 10^{-5}$

TABLE I: Decay rates  $\Gamma$  for the most stable resonance of the potential  $V(x) = \cos(x)$ , taken from Ref. [27] (CS method) and computed by the CAP grid relaxation method. Particularly for small decay rates the new CAP method proves more efficient than the CF technique.

## III. RESONANTLY ENHANCED TUNNELING

We use the CAP method to investigate how a nonlinear interaction affects the decay of a BEC in a tilted optical lattice. In the weakly interacting regime, the scaling of the decay rate with the field strength is given by the celebrated Landau-Zener formula  $\Gamma(F) \approx F \exp(\pi \Delta E^2 / F)$ , where  $\Delta E$  is the energy gap between the Bloch bands of the periodic potential [2, 14] and the field strength  $F$  determines the oscillation frequency in the bands [14, 16]. Major differences arise in the regime of RET. In this case an eigenstate localized mainly in one of the wells of the potential becomes energetically degenerate with an excited state in another well, which can increase the decay rate by orders of magnitude [16]. In the following, we focus on the experimentally studied regime [11, 12], where already a modest nonlinearity strongly affects the decay of the condensate [11, 12, 17].

RET is illustrated in Fig. 1 (a,b) for the linear case  $g = 0$ , showing the decay rate  $\Gamma$  and the chemical potential  $\mu$  of the two most stable resonances as a function of  $F$ . RET is observed at  $1/F \approx 7.5$ , where the two energy levels  $\mu(F)$  cross. The resonant coupling to the excited states leads to a pronounced RET peak of the decay rate for the most stable resonance. Coincidentally, a pronounced dip is observed for the first excited resonance, which is stabilized by the coupling to the most stable resonance [16]. The influence of a small nonlinearity is illustrated in Fig. 1 (c). Three main effects are observed: a shift of the resonance peaks, an increase (decrease) of the peak decay rate in the ground state for  $g > 0$  ( $g < 0$ ) and a deformation of the peak shape.

The shift and the deformation can be qualitatively understood by a perturbative approach [27]. To first order, this predicts a shift of the real part of the eigenenergy,

$$\Delta\mu(g) \approx g \int_{x_\ell}^{x_r} |\psi_g|^2 |\psi_{g=0}|^2 dx \approx g \int_{x_\ell}^{x_r} |\psi_{g=0}|^4 dx, \quad (5)$$

which corresponds to a shift of the field strength accord-

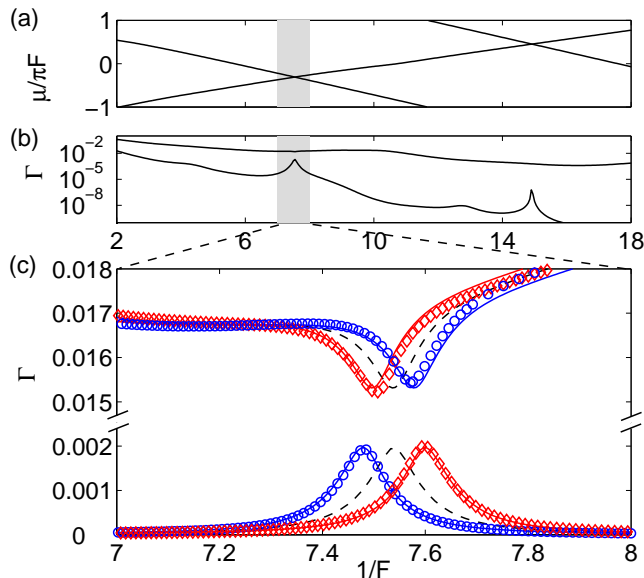


FIG. 1: (Color online) Resonantly enhanced tunneling (RET) of (non)-linear WS resonances. (a) Energies and (b) decay rates of the two most stable WS resonances in a cosine-potential as a function of the inverse field strength  $1/F$ . (c) Shift of the RET peaks due to the nonlinear interaction of a BEC for  $g = +0.02$  ( $\circ$ ),  $g = -0.02$  ( $\diamond$ ) and  $g = 0$  (- -). Numerical results (symbols) are compared to a perturbative calculation (solid lines) according to Eqs. (5) and (7).

ing to

$$\Delta F(g) \approx \pm \Delta \mu(g)/(2\pi). \quad (6)$$

Here, the minus sign holds for the ground and the plus sign for the excited band. The nonlinear decay rate is then approximately given by

$$\Gamma_g(F) = \Gamma_0(F + \Delta F(g)). \quad (7)$$

The shift is further investigated in Fig. 2 (b), where the decay rate as well as the peak position is plotted vs. the interaction strength over a wide parameter range. The perturbative calculation (5) predicts that the peak position  $F_{\text{res}}$  is shifted with a slope  $dF_{\text{res}}/dg = 0.059$  for small values of  $g$ , which is plotted as a green line in Fig. 2 (b). This deviates from the numerically exact results already for small values of  $g$ , for which a linear fit yields a smaller slope of  $dF_{\text{res}}/dg = 0.051$ . In agreement with [27] we thus find that first order perturbation theory is not sufficient to describe the shift of the RET peaks quantitatively. Noticeably, the RET peak and the dip of the decay rate for the first excited resonance always shift into opposite directions, as shown in Fig. 1 (c).

The change in the maximum decay rate is not predicted by perturbation theory, but easily explained phenomenologically. It is a direct consequence of the interaction as repulsion between the particles in general leads to a destabilization whereas attraction leads to a stabilization of both resonances and bound states (see [10] and references therein). This is further illustrated in Fig. 2 (c),

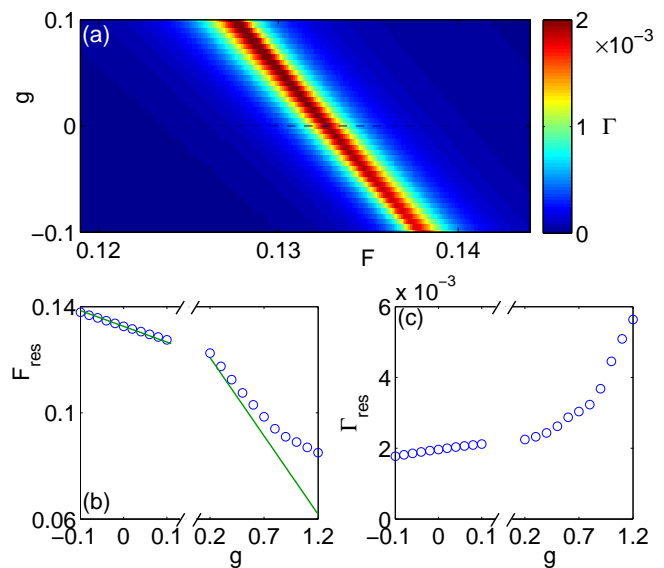


FIG. 2: (Color online) (a) Colormap plot of the decay rates of the most stable WS resonances in a cosine-potential vs. the field strength  $F$  and the interaction strength  $g$  in the vicinity of the first order RET peak. (b) Position and (c) height of the RET peak vs. the interaction strength  $g$ .

where the peak decay rate of the most stable resonance is plotted as a function of  $g$  over a wide parameter range. Similar effects have been investigated for several other model potentials [23, 25, 28].

Figure 1 (c) furthermore shows that the RET peaks become asymmetric for  $g \neq 0$ . For a repulsive (attractive) nonlinearity, the peak bends to higher (lower) values of  $F$ . If the nonlinearity is increased above a critical value  $g_{\text{cr}}$ , the peaks bend over and a bistable behavior emerges as shown in Fig. 3. For the given parameters of the optical lattice, the critical nonlinearity has been found at  $0.5 < g_{\text{cr}} < 0.6$ . The detailed shape of a bistable RET peak is plotted in the upper panel of Fig. 3, which also indicates how WS states are calculated numerically in the bistable regime: We have started with a small value of  $F$  which was then gradually increased, using every result as initial guess for the next calculation (cf. also [27, 28]). After reaching a final, large value of the field strength, the procedure was reversed and  $F$  was decreased back to the initial value. Within the regime of bistability forward and backward sweep yield the upper and lower branches of the peak, respectively. The intermediate branch is generally difficult to compute as it is dynamically unstable.

The bending of the RET peak and the emergence of a bistability can be understood qualitatively by the perturbative approach introduced above. A common WS state in a deep optical lattice is strongly localized in a single potential well so that its chemical potential is strongly changed according to Eq. (7). In comparison, the state corresponding to the maximum of the RET peak is delocalized because of the energetical degeneracy with an excited state in another well. Therefore its chemical po-

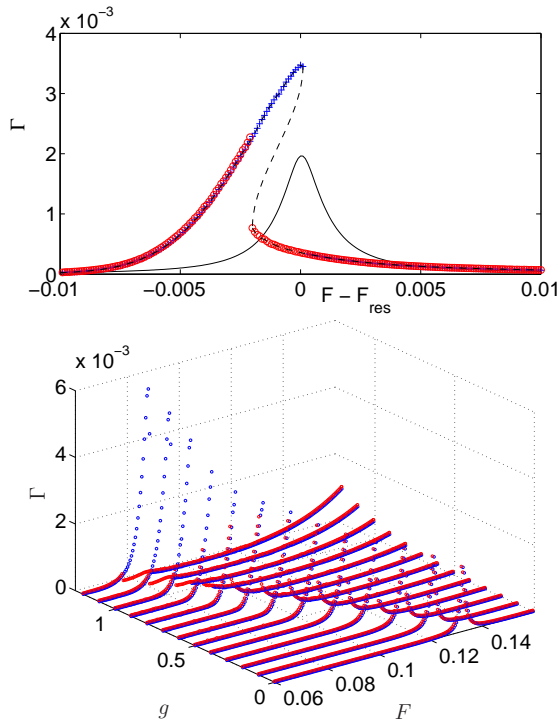


FIG. 3: (Color online) Upper panel: Bistability of the RET peak for strong repulsive interactions ( $g = 0.8$ ). The decay rate was calculated for a forward sweep (blue asterisk) and a backward sweep (red circles). A spline interpolation (dashed line) is included to guide the eye. The solid line shows the linear ( $g = 0$ ) peak shape for comparison. Lower panel: Emergence of the bistable behavior: Decay rate as a function of the field strength  $F$  for different values of the nonlinearity  $g$ .

tential is affected rather weakly and according to Eq. (6) also the change of the peak position  $\Delta F$  is small (cf. also [11, 12]). With increasing nonlinearity, the edges of a RET peak shift to smaller values of the field strength, while the maximum falls behind. The whole peak bends to the right and finally becomes bistable.

Bifurcations of nonlinear stationary states have been previously observed in several different contexts: The simplest example is the nonlinear two-mode system. As observed for the WS state, new stationary states emerge first in the vicinity of the resonance, i.e. the when the chemical potential of the two coupled states coincides (see [20, 29] and references therein). The emergence of bistability has also been analyzed for the transmission coefficient in the context of nonlinear RET through one-dimensional potential barriers [22–24]. However, in this case states corresponding to the transmission maximum are localized strongest. Thus the resonance peaks bend into the same direction as they are shifted, which is opposite from the behavior of the WS RET peaks shown in Fig. 3.

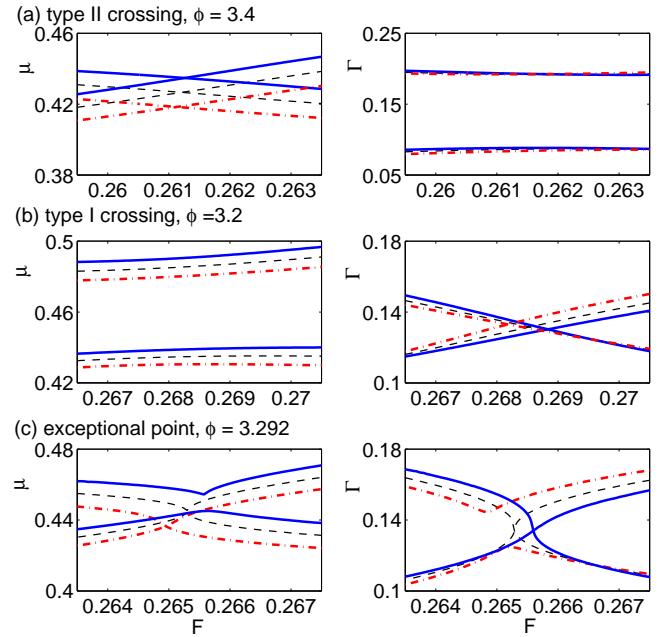


FIG. 4: (Color online) Chemical potential  $\mu$  and decay rates  $\Gamma$  for (non)linear WS-resonances in a bichromatic optical lattice for  $\delta = 1$  and (a)  $\phi = 3.4$ , (b)  $\phi = 3.2$ , (c)  $\phi = 3.292$  and  $g = 0$  (dashed black line),  $g = +0.02$  (thick blue line) and  $g = -0.02$  (dash-dotted red line).

#### IV. BEYOND THE RET-REGIME

A new regime of RET can be explored in bichromatic optical lattices,

$$V(x) = \frac{1}{2} \cos(x) + \frac{\delta}{2} \cos(2x + \phi). \quad (8)$$

These potentials can be realized experimentally by superimposing two incoherent optical lattices [33, 34], or by combining optical potentials based on virtual two-photon and four-photon processes [35, 36]. It can be used to study Landau-Zener tunneling between different bands and the interplay of tunneling and Bloch oscillations [37].

Depending on the relative phase and the relative strength of the two lattices, WS resonances in tilted bichromatic optical lattices show two remarkably different types of level crossing scenarios [38] – either the real parts  $\mu$  (type-I) or the imaginary parts  $\Gamma$  (type-II) of the eigenenergies cross. A full degeneracy of both  $\mu$  and  $\Gamma$  occurs only for isolated points in parameter space. Examples are shown in Fig. 4 for  $\delta = 1$  and different relative phases  $\phi$  of the two lattices. A familiar type-II crossing is observed for  $\phi = 3.4$ . The real parts of the eigenenergies ( $\mu$ ) cross, while the imaginary parts  $\Gamma$  anti-cross, leading to the familiar RET-peaks of the decay rates. Changing the phase slightly to  $\phi = 3.2$ , one finds a type-I crossing. The decay rates of the two most stable resonances now cross, while the *real* parts anti-cross. An exceptional point [38], where both real and imaginary part are fully degenerate, is found for  $\phi = 3.292$ , as shown in Fig. 4

(c). However, the degeneracy is lifted as soon as the atoms start to interact. A weak repulsive nonlinearity  $g = +0.02$ , turns the exceptional crossing into an ordinary type-I crossing, while an attractive nonlinearity  $g = -0.02$  favors a type-II crossing. This change of peak shape can have dramatic effects on the dynamics of a Bose-Einstein condensate, in particular when experimental parameters are adiabatically varied (see, e.g. [38]).

## V. CONCLUSIONS

Bose-Einstein condensates in tilted optical lattices are ideal to study the decay of interacting open quantum systems. Experimentally the parameters can be tuned over a wide range and the dynamics can be recorded in situ. Here we presented an *efficient* method to calculate the decay rate in the mean-field regime also in the presence of degeneracies which also, unlike previous methods, is not restricted to ground state calculations. The effects of the nonlinearity are strongest in the regime of reso-

nant tunneling, where the decay rate can be enhanced by orders of magnitude by resonant coupling to unstable excited states. The interactions shift and bend the resonance peaks and eventually lead to a bistable peak shape. Even more interesting effects can be studied in tilted bichromatic lattices, where different types of level crossing scenarios emerge when the lattice parameters are tuned. These effects will be studied in detail in a future publication.

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